IB DP Mathematics (AA HL)
Internal Assessment
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Modelling the shape of Jelly Beans and using Volumes of Revolution to estimate the No. of Jelly beans in a Jar for a Candy Guessing Contest

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Table of Contents

I. INTRODUCTION

Candy guessing games offer numerous awards ranging from candy packets, amazon gift cards/vouchers, and soft toys for the participant whose guess is the closest to the actual value of the number of candies. Regardless of these incentivizing benefits many participants still opt for a wild guess and rely on their 'luck' to win the game instead of making an educated guess. A more logical approach would be to use modelling and volumes of revolution of jelly bean instead. This investigation will explore the number of candies in the jar (figure 2) using volumes of revolution.

Table 1 Estimates of the number of jelly beans inside this jar by a random sample of 35 people

Estimated No. of Candies						
500	520	544	812	350	77	143
730	187	625	329	2000	675	709
602	439	502	200	1500	250	864
393	900	587	742	895	556	693
452	243	320	342	1015	115	100
MEAN ($\overline{x}$) of this Sample					563	
STANDARD DEVIATION (σ)					390	

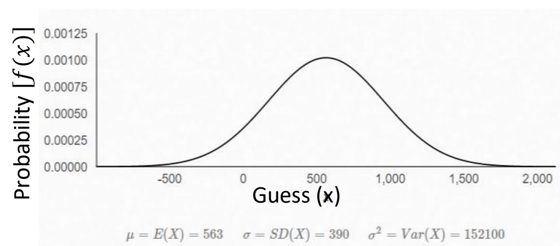

Figure 1 Normal Distribution Bell Curve for Sample Data (Candidate, 2021)

After surveying a random sample of 35 people through google forms (refer to Appendix A) at the Candy Guessing Contest hosted on International Day at my school (using Google Forms), the average estimate of the number of jelly beans inside this jar (in figure 2) was 563 (see table 1). As an active and eager participant, I also took my own wild guess of 500 based on my instincts. According to the data collected in the survey (on google forms, please refer to the pie chart in **figure 3**), 80% of the people took a wild guess based on their instincts to guess the no. of candies in a jar, which is not only illogical but also absurd because they reach their estimations without a rational approach. This sample of guesses has a really high standard deviation (of 390: **figure 1**) which supports that idea that wild guess and random estimation is highly inefficient and unreliable. Still, this approach is widely taken, for its simplicity and time efficiency. Whilst only 12% of the people used a volume of a sphere

Figure 2 An image of a sample jar of Jelly beans (Self-taken, 2021)

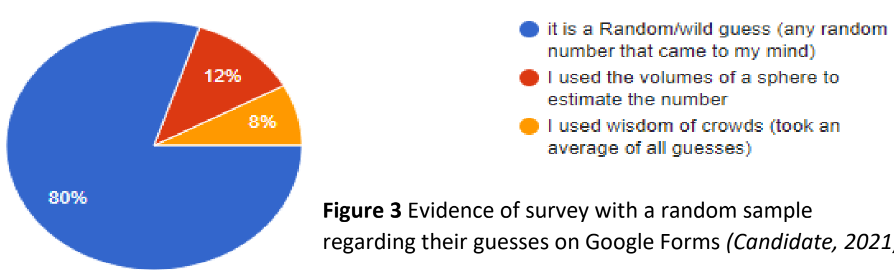

Figure 3 Evidence of survey with a random sample regarding their guesses on Google Forms *(Candidate, 2021)*

and several other assumptions to form a guess. Although, these assumptions yield a result slightly more accurate and probable value in comparison to a wild guess. This assumption does not account for the egg-like shape of a jelly bean which causes the packing density to vary. A more reliable method however would be to use the volumes of revolution to yield a more accurate value for the volume of the jar and jelly beans and therefore reduce the margin of error from the actual no. of jelly beans. However, with a correct approach and proper calculations, the chances of winning this game can increase. The standard deviation and variance for this sample data is very high which suggests that the guesses taken by people are absolutely random and absurd because of the high degree of spread of data. This normal distribution curve (in **figure 1** based on random guesses) will be then compared to the normal distribution curve plotted using the estimates (of 35 people) from the number of candies derived using modelling and volumes of revolution of Jelly Beans.

Pre-existing Methods: Before using volumes of revolution to estimate the no. of candies in a guessing game it is important to acknowledge the limitations of existing methods of estimation.

Table 2 Pre-existing methods of Guessing (*"Sampling and Estimating: How Many Jellybeans?"*)

Method	Description	Limitation
The Wisdom of Crowds ("The Right Way to Use the Wisdom of Crowds.")	This method relies on the concept that many are smarter than a few. It basically relies on the diversity prediction theorem which states that the average guess/prediction of the crowd is far more accurate than your own guess. Hence, in these candy contests, one must average out the guesses of other participating candidates.	- Small Sample – If you average out the guesses of a small sample of individuals from all participants, the margin of error will be greater and hence your prediction will be far off the true value. - In extremely competitive contests, participants do not share their predictions and hence this method is ineffective. - Still dependent on random guess which is less accurate than volumes of revolution.
Estimation by mass	This method of estimation involves measuring the mass of a single candy on a weighing scale and measuring the mass of the jar occupied with the jelly beans and without the jelly beans. After finding the mass of jelly beans within the jar, the total volume will be divided by the mass of a single jelly bean to predict the no. of jelly beans.	- Average uncertainty of an accurate electric weighing balance is ±0.1 g. Although this is extremely accurate relative to one jelly bean, the total uncertainty in the mass will be extremely high when considering a jar containing more than 100 candies and therefore the overall uncertainty is significantly high. - Theoretically, participants do not have the right to measure the mass of the jar within this game as it may provide an unfair advantage.

Due to the limitations aforementioned, I believe that using volumes of revolution would yield to a far more accurate estimate of the number of candies inside the jar (shown in **figure 2**).

Since these contests offer sufficient rewards, doing a little homework can certainly provide you with a significant advantage over the other competing candidates. The true number of jelly beans in this sample jar will only be revealed after modelling the shape of the jelly bean and using the packing efficiency and volumes of revolution to examine the effectiveness of this method.

III. RATIONALE

Figure 4 Carnival Guessing Booth at my School's Winter Souk before the Pandemic (Self-taken, 2018)

Due to the inborn nature of my sweet tooth and becoming a fan of candies, I am always eager to participate in Candy guessing games from the age of 7; however, I have never won a single one myself. It is certainly amusing in retrospect, that I thought I could win a game just by taking a random guess. Growing up out of curiosity, I then started organizing candy guessing games on Children Carnival Day in my community, School Winter Souk (see evidence **Figure 4** on the right) and International Day Event *(refer to Appendix B for more evidence)*. Hosting these events I realized that there are numerous other methods to guess the number of candies (including the wisdom of crowds and estimation by mass which will be explained later in Table 2) however I noticed that these methods never yield an accurate result and cannot be implemented in Carnival games because of their limitations (acknowledged later in Table 2). Therefore, I seized the Math Internal Assessment as an opportunity to investigate the number of jelly beans in the jar (**figure 2**) using volumes of revolution. From my experience of organizing candy-guessing games, the winner was always off from the true value. As a result, I wanted to explore how using the packing efficiency and volumes of revolution can allow individuals to make an 'Educated Guess' and actually yield a more accurate value by calculating the percentage error. The strategy of using the packing efficiency (the ratio of occupied volume to the total volume of the container) of candies has been proposed to be effective at yielding more reliable results.

However, this is only accurate for uniform-spherical hard candies. But, I wanted to use this method of approach for something more irregular like jelly beans because they are far more popular in contests than spherical-shaped candies. Because of their irregular shape, participants are more reluctant to use this method of packing efficiency thinking that there will still be a large margin of error from the real value. To make this approach more reliable for jelly beans, with their irregular shape, I want to investigate a slightly extended strategy where modeling the shape using a function and estimating the volume help candidates reach a more accurate number with a smaller margin of error and possibly win this carnival contest.

IV. AIM

The aim for my exploration is to use a different method of estimation (i.e. volume of revolutions) to determine value of the predicted no. of candies inside the jar in **figure 2** and determine its accuracy (relative to the real value), instead of taking an absurd wild guess to win a candy guessing game in this contest. This method of estimation will rely on the volume of revolution of both: the jar and the jelly beans.

V. MATHEMATICAL METHOD

*__Reflection:__ For the sake of precision and consistency all measurements for dimensions of the jar and the jelly bean are taken in millimeters using a vernier caliper (± 0.01 mm) and graphed accordingly on GeoGebra using a millimeter scale on the X-axis and Y-axis. For consistency and accuracy all values for the volume of revolution have been quoted to three decimal places to ensure that the rounding does not impact the final estimate of the number of candies in the jar.

The mathematical method is divided into three steps as follows:

1. **Part 1:** Determining the Packing Efficiency of Jelly Beans
2. **Part 2:** Modelling the shape of the Jar and Calculating its Occupied Volume.
3. **Part 3:** Modelling the shape of a jelly bean and estimating the no. of jelly beans in the jar

PART 1: Determining the Packing Efficiency of Jelly Beans

Since the jelly beans do not completely occupy the complete volume of the container it is important to take into consideration the packing efficiencies of the candies. **Packing efficiency** (or packing density) of a particular figure/shape describes the fraction of the space occupied by that object in a particular packing container ("Packing Density."). This is usually used to determine the fraction of space filled by objects that have air pockets and are arranged randomly by chance. This will help account for the empty space between candies.

To determine the packing efficiency of the jelly beans, I assumed that they are shaped like a sphere, because a spheroid is a form of a deformed sphere, therefore there will be a negligible amount of difference between the packing efficiency of the sphere and a spheroid. This assumption is made because the packing efficiency for a spheroid will vary based on the optimum configuration and the lateral or vertical arrangement. Hence for the sake of simplicity of calculation of the packing efficiency the shape is assumed as a sphere with a radius r.

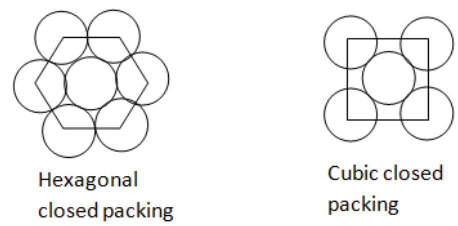

Figure 5 *Hexagonal Vs. Cubical Closed Packing (“Distinguish between Hexagonal Close Packing and Cubic Close Packing.”)*

Looking at the random arrangement of a spherical body, spheres can either arrange themselves in a cubical closest packing or hexagonal closest packing (see **figure 5** above). However, in a random packing, hexagonal closest packing is more commonly observed in nature because of an optimal conformation. Therefore to determine the packing efficiency, hexagonal closed packing is assumed throughout the jar. Because the arrangement is a hexagonal closed packing, I have used a hexagonal prism as a unit cell to find air gaps in between the spheres (see figure 4 on the left).

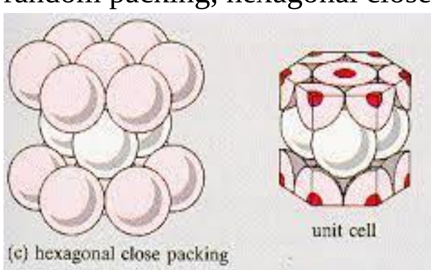

Figure 6 *Hexagonal Closed Packing and Hexagonal Unit Cell (“Hexagonal Close Packed.”)*

The packing efficiency will be calculated using the formula

$$Packing\ efficiency = \frac{Total\ Volume\ occupied\ by\ spherical\ sectors}{Volume\ of\ the\ unit\ cell\ (hexagonal\ prism)} \qquad \textbf{(1)}$$

Separating the planes of the hexagonal closed packing inside the unit cell shown in **figure 6** will produce **figure 7** with sections A, B and C. Section C is the reflection of the base of the unit cell.

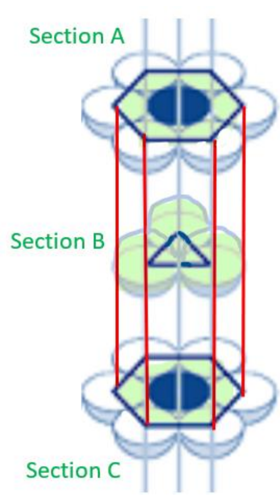

Figure 7 *Sectional drawing of the packing arrangement (Candidate, 2021)*

The total volume covered ($V_{(Total)}$) by the spherical sector enclosed inside the hexagonal unit cell is equal to the sum of spherical sectors (shown in green) in sections A, B and C where r is the sphere radius.

$$V_{(Section\ A)} = V_{Hemisphere} + 6\left(\frac{1}{3}V_{Hemisphere}\right) = \frac{3}{2}V_{Sphere} = 2\pi r^3 \qquad \textbf{(2)}$$

The spherical segments at the corner of the hexagonal prism occupy of a volume $\frac{1}{3}V_{Hemisphere}$ each because of the 120° angle inside the hexagon which is exactly $\frac{1}{3}$ of a complete revolution.

$$V_{(Section\ B)} = 3V_{Sphere} = 3\left(\frac{4}{3}\pi r^3\right) = 4\pi r^3 \qquad \textbf{(3)}$$

$$V_{(Section\ C)} = V_{(Section\ A)} = 2\pi r^3 \qquad \textbf{(4)}$$

$$\therefore\ V_{(Total)} = V_{(Section\ A)} + V_{(Section\ B)} + V_{(Section\ C)} = 8\pi r^3 \qquad \textbf{(5)}$$

Calculating the volume of the Unit Cell (Hexagonal Prism)

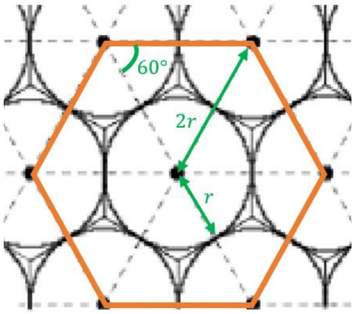

Figure 8 Hexagonal Base Area (Candidate, 2021)

$$Cross - sectional\ Area\ of\ the\ Prism = 6\left(\frac{1}{2}(2r)^2 Sin\ 60\right) \qquad \textbf{(6)}$$

where r is the radius of the sphere.

A factor of 6 is placed because there are 6 equilateral triangles inside the hexagonal base of the prism (refer to **figure 8**).

$$\therefore Cross - sectional\ Area = 6\sqrt{3}r^2 \qquad \textbf{(7)}$$

Finding the height of the hexagonal prism

To find the height of the hexagonal prism, a tetrahedron is constructed inside the central spheres by connecting their centres with each other (see **figure 9 (a) and 9 (b)** on the following page). The height of the unit cell (hexagonal prism) is equivelent to the cumulative height of two tetrahedrons (as shown in **figure 9(a)**). To determine the height of the tetrahedron (x), we must apply the Pythagoras theorem in a 3D space.

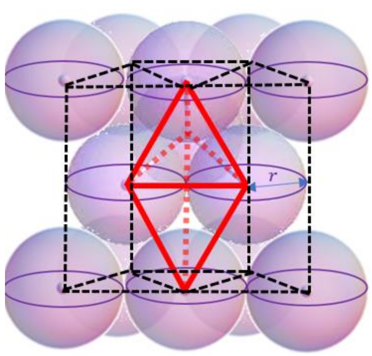

Figure 9 (a) *Horizontal cross-section of the first three layers of hexagonal packing (Candidate, 2021)*

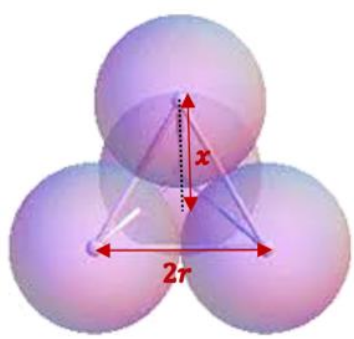

Figure 9 (b) *Tetrahedron formation of the inner three spheres inside the hexagonal closed packing structure (Candidate, 2021)*

Considering the base of the tetrahedron (see **figure 9** to the right), $\overline{DG}$ and $\overline{CF}$ are angle bisectors. Therefore angle $C\hat{F}D$ measures 90°, $F\hat{D}G$ measures 30°, and line segment $\overline{DF}$ measures a distance r due to angle bisectors properties within an equilateral triangle.

With the information provided, we can calculate the distance of the line segment $\overline{DE}$ using trigonometric identities within a right-angled triangle.

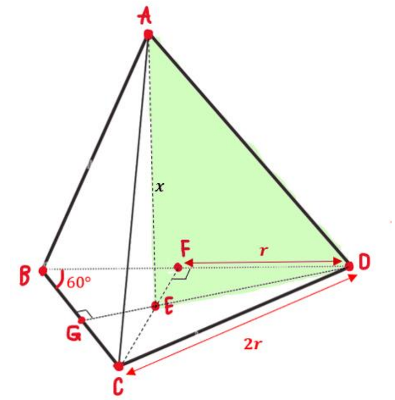

***Figure 9** Annotated Diagram of a Tetrahedron (Candidate, 2021)*

$$Cos(30) = \frac{r}{\overline{DE}} \quad \textbf{(8)}$$

$$\overline{DE} = \frac{r}{Cos\,(30)} = \frac{2r}{\sqrt{3}} = \frac{2\sqrt{3}r}{3} \quad \textbf{(9)}$$

To calculate the height of the tetrahedron, we must consider the plane in green (inside the tetrahedron in figure) with the points A, E & D. Since point E is the centre of the base of the tetrahedron, the triangle enclosed by point A, E and D would be a right angle triangle.

It is important to also note that line segment $\overline{AD} = 2r$, as the line segment connects the two centers of adjacent spheres. With the assistance of the Pythagoras theorem and substituting the measurements previously derived for relevant line segments ($\overline{AD}$ and $\overline{DE}$), the height (x or $\overline{AE}$) can be determined.

$$\overline{AE}^2 + \overline{DE}^2 = \overline{AD}^2 \quad \textbf{(10)}$$

$$x^2 + \left(\frac{2\sqrt{3}r}{3}\right)^2 = (2r)^2 \quad \textbf{(11)}$$

$$x^2 + \frac{4r^2}{3} = 4r^2 \quad \textbf{(12)}$$

$$x^2 = \frac{8r^2}{3} \quad \textbf{(13)}$$

$$\therefore x = \frac{2\sqrt{6}}{3}r \quad \textbf{(14)}$$

For <u>equation 14</u>: Only positive value of x is taken into consideration because $x > 0$ since the height of an object cannot be negative.

Therefore the height of the prism (equivalent to the height of two tetrahedrons) is given by

$$h = 2x = \frac{4\sqrt{6}}{3}r \quad \textbf{(15)}$$

Hence we can find the volume of the unit cell (hexagonal prism)

$$V_{(Unit\ Cell)} = Base\ Area\ \times Height = 6\sqrt{3}r^2\left(\frac{4\sqrt{6}}{3}r\right) = 24\sqrt{2}r^3 \quad \textbf{(16)}$$

Determining the packing efficiency of the Jelly beans

$$Packing\ efficiency = \frac{V_{(Total)}}{V_{(Unit\ Cell)}} = \frac{8\pi r^3}{24\sqrt{2}r^3} \approx 0.74048 \text{ (5 d.p.)} \qquad \textbf{(17)}$$

Reflection – The packing efficiency is quoted to 5 d.p. to ensure that rounding does not influence volume calculations. It is important to note that the literature value for random packing of an ellipsoid is 0.7585 which means that the value I derived as a percentage error of 2.37 % which is miniscule but has occurred due to assumptions made in the derivations. Furthermore it is important to note that this packing efficiency assumes that jelly beans arrange themselves in the optimal hexagonal configuration which is not entirely possible because of the deformities in the jelly beans which can increase the percentage error.

PART 2: Modelling the Occupied Volume of the Glass Jar

The sample glass jar used in this contest (refer to **figure 2**) was 11 cm by 22 cm. Converting this dimension into millimeters for consistency equals 220 $mm \times 110\ mm$. To find the total volume of the jar, GeoGebra was used to model the equation $f(x)$ of the glass jar to a 6th degree polynomial (**figure 10**).

Reflection – Although a sixth degree polynomial is not completely accurate with some excess at the sides of the curve it is relatively more reliable in comparison to assuming a cylindrical shape. It is important to note that a higher degree polynomial would yield a more accurate function to model the jar, it was not done for the sake of simplicity of the formula.

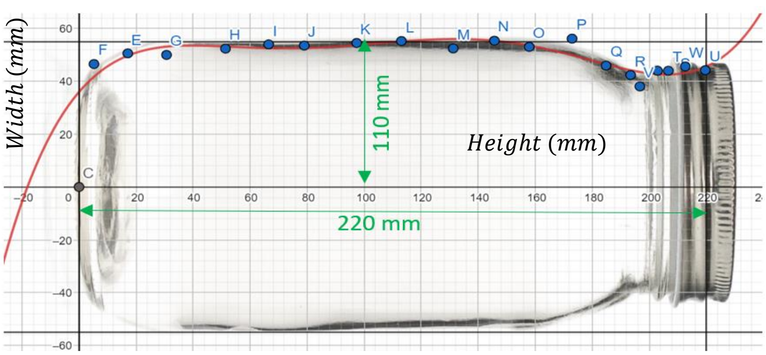

Figure 10 Empty Glass Jar modelled by the function f(x) on Geogebra (Candidate, 2021)

The overall function (polynomial fit) of the jar was obtained to be:

$$f(x) = -0.0000000000008\,x^6 + 0.000000003663\,x^5 - 0.0000019004691\,x^4 + 0.0003726156686\,x^3 - 0.0326655698788\,x^2 + 1.2619820939614\,x + 36.0119953845667 \qquad \textbf{(18)}$$

Limits: $(0\ mm < x < 220\ mm)$

***IN CONTEXT OF CANDY GUESSING GAMES:** Since the same sized jar with the same dimensions is not available in all candy guessing games it is important to note that this method of modelling can be adapted for any type and shape of jar. The same approach will be utilized to model the shape of the jar with an nth degree polynomial just with a different function.

This curve will be rotated by $\mathbf{360°}$ across the x-axis to obtain the volume of revolution given by:

$$Volume\ of\ Revolution = \pi\int_b^a [f(x)]^2\,dx \qquad \textbf{(19)}$$

The upper (a) and the lower (b) limit will represent the length of the jar hence 220 mm and 0 mm respectively. The volume of revolution for the graph of was created by 360 degrees along the x-axis (see **figure 11** on the right).

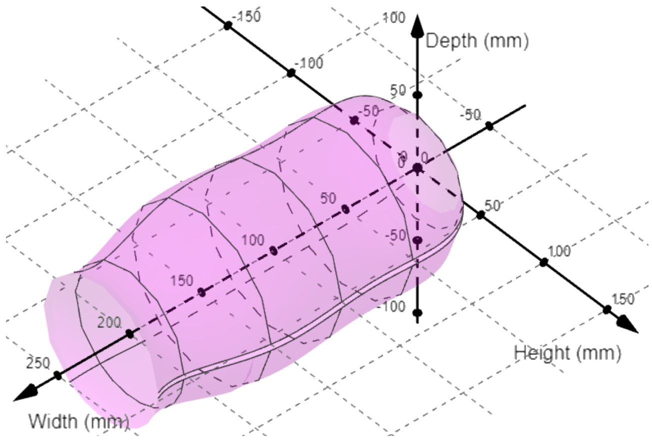

Figure 11 Volume of revolution of the glass jar on Geogebra (Candidate, 2021)

The total volume of the jar is obtained to be $1812452.393\ mm^3$ which is equivalent to $1812.452\ cm^3$. The volume quoted on the label of the jar is 2 litres which are equivalent to 2000 cm^3. The percentage error can be calculated by using the formula:

$$Percentage\ error = \left|\frac{Literature\ Value - Obtained\ Value}{Literature\ Value}\right| \times 100 \qquad \textbf{(20)}$$

Therefore the percentage error for the volume of the jar is:

$$Percentage\ error = \left|\frac{2000\ cm^3 - 1812.45\ cm^3}{2000\ \ cm^3}\right| \times 100 = 9.38\ \% \qquad \textbf{(21)}$$

Reflection – A percentage error of 9.38 percent appears to be significantly large however this is due to the fact the volume of the lid was removed. Because the jelly beans are not completely filled to the top part, there is a percentage difference between the two values. It is also worth knowing that there are alternative faster methods to measure the volume of the jar by filling the jar with water and measuring the volume of water after draining it into a measuring cylinder. This is feasible but is normally not allowed in such contests as gives an unfair advantage to particular candidates.

Using a n^{th} degree polynomial to model the shape is accurate for regular shaped jar, but jars that have a really fancy shape will contain a lot of excess or unoccupied area. Therefore for fancy jars it is important to consider that the volume obtained by the volume of revolution will be less reliable with a greater degree of uncertainty.

For jelly beans, the value of packing efficiency that I have used is 0.74048 (derived in Part 1).

Table 3 Nomenclature of Variables

Symbol	Definition
V_O	Volume occupied by the jelly beans
V_T	Total Volume of the Container
γ	Packing Efficiency *(derived in Part 1)*
V_J	Volume of a single jelly bean

Reflection – The value of packing density is not an exact estimate and different studies have quoted different values for spheroids. Assuming the value of the packing fraction can cause uncertainty in the final obtained value for the estimate of the no. of Jelly beans.

Hence, the volume occupied by jelly beans is assumed to be equal to:

$$V_O = . V_T \gamma \quad \textbf{(22)}$$

Substituting the values of the total volume and the packing density the occupied volume equals:

$$V_O = (1812452.39\ mm^3) \times (0.74048\ ) = 1342084.746\ mm^3 \quad \textbf{(23)}$$

The values have been quoted to three decimal places to ensure that the rounding of the last few digits does not increase the uncertainty of the final answer.

PART 3: The shape of a single Jelly Bean and Estimating the no. of Jelly beans

Figure 12 Venier Calliper Readings (Candidate, 2021)

Preliminary Measurements of the Dimensions of the Jelly Bean: In order to estimate the volume of a single jelly bean, I needed to model the shape of a single jelly bean using a graphical function. So before graphing the jelly bean and finding the volume of revolution of jelly bean, I determined the dimensions of a Jelly Bean to scale the graph. The dimensions for the sample jelly bean were averaged among 9 jelly beans to account for any irregularities and anomalies between their shapes and get a more reliable value for their size. The average dimensions were measured using a vernier caliper (± 0.01 mm), see **figure 12**, for its small uncertainty and precise values which will increase the accuracy of measurements. See average values in **Table 4**.

Table 4 Dimensions measured using a Digital Vernier Caliper (± 0.01 mm)

Jelly Bean No.	Width (mm)	Length (mm)
1	12.62	16.62
2	12.26	15.26
3	13.13	14.13
4	11.95	17.95
5	12.87	16.87
6	12.23	17.23
7	12.93	14.83
8	11.33	16.43
9	11.38	17.87
Mean ($\bar{x}$)	12.30	16.35

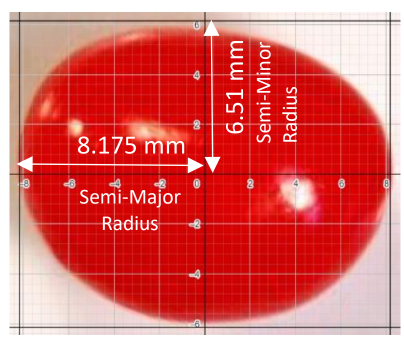

Figure 13 Scaled Diagram of the Jelly Bean on Geogebra (Candidate, 2021)

I have scaled diagram (figure 13) of the Jelly Bean on the GeoGebra using the lengths of the semi-major and semi-minor axis.

The radius of the semi-major axis is $\frac{16.35}{2} = 8.175\ mm$

The radius of semi-minor axis is $\frac{12.30}{2} = 6.15\ mm$

<u>NOTE:</u> Although jelly beans cannot have a negative length, it has been scaled accordingly on the graph to ensure that the center of the jelly bean (i.e. its center of mass) aligns with the origin of the graph for consistency and to preserve the symmetry of the jelly bean shape across the axes for finding the volume of revolution.

Method A: Fourth Root Function

I first modelled the shape of this jelly bean using a fourth root function, defined by $g(x)$. I derived $g(x)$ by manually adjusting the coefficient (vertical stretch by a scale factor of $\frac{6}{\sqrt{3}}$ on Geogebra.

$$g(x) = \frac{6}{\sqrt{3}} \sqrt[4]{8.175 - x} \qquad \textbf{(24)}$$

$g(x)$ only models the jelly bean in the first quadrant where $0\ mm < x < 8.175\ mm$.

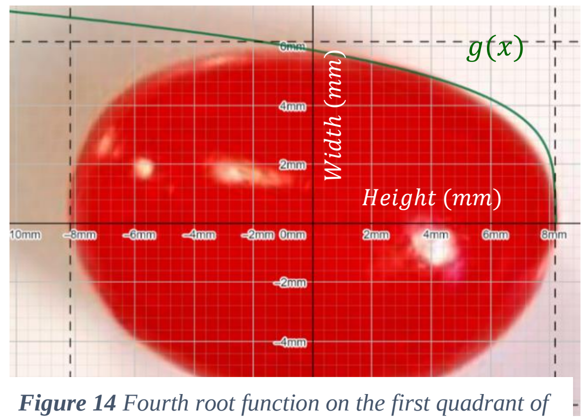

Figure 14 Fourth root function on the first quadrant of the jelly beans (Candidate on GeoGebra, 2021)

In order to calculate the volume of the jelly bean using this function, the volume will be calculated between the bounds x = 0 and x = 8.175 (in the first quadrant only, refer to **figure 14**) using the volume of revolution about the x-axis. This function ($g(x)$) was substituted into the volume of revolution formula. The upper bound a is equal to 8.175 and lower bound b is equal to 0.

Reflection – After reflecting the graphs in 3 other quadrants, it is clear that there is a lot of excess part of the jelly bean that is not included by this function (as seen above the green line representing the graph of g(x)) which means that this function will yield a lot of discrepancies from the real approximation of the volume of this jelly bean.

$$V_J = \pi \int_b^a [g(x)]^2\,dx = \pi \int_0^{8.175} \left[\frac{6}{\sqrt{3}} \sqrt[4]{8.175 - x}\right]^2 dx$$
$$= 587.451\ mm^3 \qquad \textbf{(25)}$$

Evaluating the integral above, the value obtained for a volume of a half jelly bean is $587.45\ mm^3$. This obtained value is then multiplied by a factor of 2 to take into consideration the other side of the jelly bean opposite to the first quadrant (i.e. the left hemisphere). Therefore the total volume equals:

Reflection – This value for the volume of the solid (jelly bean) is extremely large for it to be true. Therefore modifying the function to model the jelly beans is apparent to be more accurate.

$$V_J = 2 \times 587.45\ mm^3 = 1174.901\ mm^3 \qquad \textbf{(26)}$$

Hence the total no. of jelly beans in this sample can be approximated to equal:

$$No.of\ Jelly\ Beans = \frac{V_O}{V_J} = \frac{1342084.746\ \ mm^3}{1174.901\ mm^3} = 1142.295 \qquad \textbf{(27)}$$

This value is rounded off to 1142 because no. of jelly beans can be an integer value only. Since this number seems too low for this size of jar, presumably because the function did not fully cover the boundaries of the jelly bean precisely, it is important to use a different function.

To observe the complete model of jelly bean this graph is reflected across the y-axis and x-axis as shown in blue in **figure 15.** The first quadrant is modelled using $g(x)$. The remaining quadrants are modelled by the following functions:

Table 5 Reflections of $g(x)$

Second Quadrant	Third Quadrant	Fourth Quadrant
$a(x) = \frac{6}{\sqrt{3}}\sqrt[4]{8.175 + x}$	$c(x) = -\frac{6}{\sqrt{3}}\sqrt[4]{8.175 + x}$	$c(x) = -\frac{6}{\sqrt{3}}\sqrt[4]{8.175 - x}$

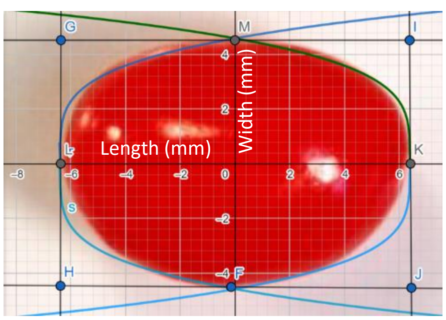

Figure 15 Reflections of the fourth root function on GeoGebra (Candidate, 2021)

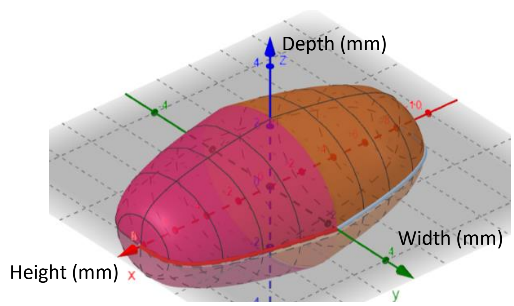

Figure 16 3D volume of revolution diagram of the forth root function and its reflection about the y-axis on Geogebra (Candidate, 2021

Method B: Ellipsis Function

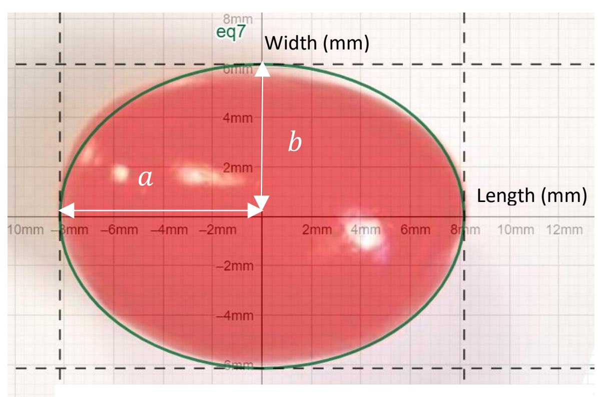

Figure 17 Ellipse Function modelling a Jelly Bean on GeoGebra (Candidate, 2021)

Since the model derived in Method A is not very accurate, the function was changed into an ellipsis because it better resembles a Jelly Bean shape. Ellipsis Function can make the value slightly more reliable because the shape of a jelly bean is a lot more like a spheroid. Hence the new equation to model this shape is the general function of an ellipsis:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad \text{(“College Algebra: Equation of Ellipses.”)} \qquad \textbf{(28)}$$

where a is the radius of the semi-major axis ($8.175\ mm$) and b is the radius of the semi-minor axis ($6.15\ mm$) found using preliminary measurements of the jelly bean and (h, k) are the coordinates of the center of the ellipse. Since the center of the jelly bean lies at the origin at $(0, 0)$, h and k can be eliminated.

$$\therefore \frac{x^2}{8.175^2} + \frac{y^2}{6.15^2} = 1 \tag{29}$$

Using the ellipsis function the new graph is produced in **figure 17**.

Equation 29 above has been rearranged to make y^2 the subject of the formula which can easily be substituted into the volume of revolution equation.

$$y^2 = 6.15^2\left(1 - \frac{x^2}{8.175^2}\right) = 6.15^2 - \frac{6.15^2x^2}{8.175^2} \tag{30}$$

Substituting the equation 30 y^2 into the volume of revolution equation gives:

$$V_J = \pi \int_b^a [y]^2\, dx = \pi \int_{-8.175}^{8.175} [6.15^2 - \frac{6.15^2x^2}{8.175^2}]\, dx = 1295.169\, mm^3\ (3\, d.p.) \tag{31}$$

Where a and b are the roots of the function at (-8.175,0) and (8.175, 0). The bounds reflect the magnitude of the length of the semi-major axis. The value of volume of revolution is rounded to 3 decimal places to ensure that the rounding does not affect the accuracy of the final estimate of jelly beans.

Reflection– This function seems to be more accurate than the previous one, but it also has some excess area included in the revolution above the jelly bean and also to the side, after the second root of the function. Because of its spheroid shape, the margin of error is still higher therefore it is important to consider using an modified version of an ellipsis function (with another constant) which will fit the boundaries more accurately.

Therefore the total no. of jelly beans predicted by this function is:

$$No. of\ Jelly\ Beans = \frac{V_O}{V_J} = \frac{1342084.746\ \ mm^3}{1295.169\, mm^3} = 1036.224\ (3\, d.p.) \approx 1036\ jelly\ beans \tag{32}$$

Reflection – The final value is rounded down to 1036 jelly beans It is important to note here that, changing the function to an ellipsoid causes a significant change in the number jelly beans by more than 25. This suggests that this approach will only reflect a value close to the expected no. of jelly beans provided that a correct and accurate function is used to model the shape of the jar and the

Because the first ellipsoid function did not completely enclose the entire boundary of the jelly bean. To better resemble the shape of an oval-shaped jelly bean the function was slightly modified by adding a new constant term into the denominator and is written in the general form:

$$\frac{x^2}{(a+bx)} + \frac{y^2}{c} = 1, \text{ where } \frac{-b}{a}, c \neq 0 \text{ and } b > 0 \qquad \textbf{(33)}$$

To determine the values of the three parameters a, b and c (in equation 33) systems of three linear equations is required and therefore three coordinates are required from the jelly bean model ("*Equation of Egg Shaped Curve*"). I first translated the jelly bean curve by 0.825 units such that the maximum width (point of deformation of the jelly bean) was in line with the y-axis (see **figure 18**) . This was done to ensure that the oval deformation lies on exactly on the y-axes. To simplify the process of calculations I will be using the x and y-intercepts to eliminate and x and y part of the equation above.

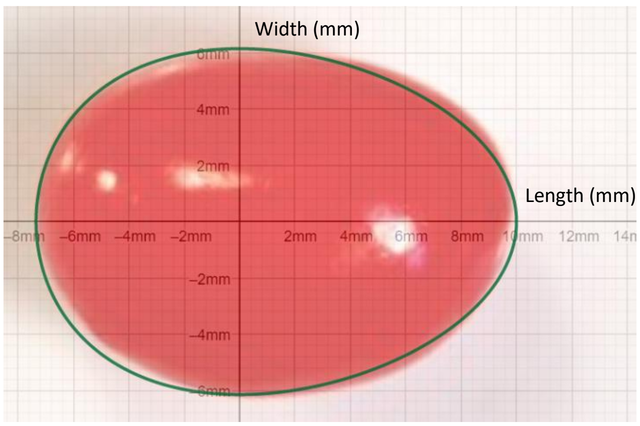

Figure 18 *Ellipse Equation with a Constant Term on GeoGebra (Candidate, 2021)*

By examining the model of translated jelly bean on GeoGebra we can find the axes intercepts:

- x-intercepts at $(10, 0)$ and $(-7.35, 0)$
- y-intercepts at $(0, 6.15)$ and $(0, -6.15)$

Substituting the values of the axes intercepts into equation 33 gives

$$\frac{6.15^2}{c} = 1, \quad \text{hence } c = 6.15^2 \qquad \textbf{(34)}$$

$$a + 10\,b = 10^2 \qquad \textbf{(35)}$$

$$a - 7.35\,b = 7.35^2 \qquad \textbf{(36)}$$

Simultaneously solving equation 35 and 36 gives $a = 73.5$ and $b = 2.65$.

Hence the equation of the oval-like shape of the jelly bean equals.

$$\frac{x^2}{(73.5+2.65x)} + \frac{y^2}{6.15^2} = 1 \qquad \textbf{(37)}$$

This new form of the function further resembles the spheroid shape of jelly beans far more precisely than the previous ellipsoid function (see the 3D diagram).

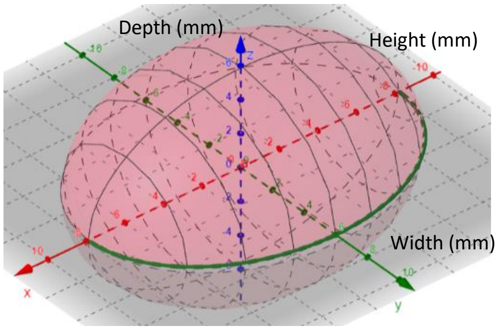

Figure 19 3-Dimensional Graphic View of a Revolution an ovoid on GeoGebra (Candidate, 2021)

Rearranging the equation 37 to make y^2 the subject

$$y^2 = 6.15^2(1 - \frac{x^2}{(73.5+2.65x)}) \quad \textbf{(38)}$$

The new roots of this equation are (10, 0) and (−7.35, 0), which will be the bounds for the volume of revolution.

Using equation 38 and its roots as upper and lower limits, the volume equals

$$V_J = \pi \int_b^a [y]^2 \, dx = 6.15^2 \, \pi \int_{-7.35}^{10} \left(1 - \frac{x^2}{(73.5+2.65x)}\right) dx \quad \textbf{(39)}$$

To evaluate this integral (equation 39) , we can use integration by substitution:

$$u = 73.5 + 2.65x \quad \textbf{(40)}$$

$$\frac{du}{dx} = 2.65 \qquad \therefore dx = \frac{du}{2.65} \quad \textbf{(41)}$$

Expressing the upper and lower limits of x in terms of u using equation 40, V_J can be rewritten as

$$V_J = \frac{6.15^2}{2.65} \, \pi \int_{54.02}^{100} \left(1 - \frac{(u-73.5)^2}{2.65^2 \, u}\right) du \quad \textbf{(42)}$$

Simplifying equation 42 above

$$V_J = \frac{6.15^2}{2.65} \, \pi \int_{54.02}^{100} \left(-\frac{u}{2.65^2} + 54.47 - \frac{73.5^2}{2.65^2 \, u}\right) du = 1367.951 \, mm^3 \quad \textbf{(43)}$$

Using this volume of a jelly bean the total no. of jelly beans equals:

$$No. of\ Jelly\ Beans = \frac{V_O}{V_J} = \frac{1342084.746 \ \ mm^3}{1367.951 \ \ mm^3} = 981.091\ (3\ d.p.) \approx 981\ jelly\ beans \quad \textbf{(44)}$$

Reflection – All values have been quoted to 3 decimal places to maintain the precision of values. In comparison to the previous function ellipsis function, this equation yields a slightly larger value for the no. of jelly beans. A difference of 55 jelly beans relative the previous is certainly capable of swaying this game towards a different winner.

Method C: Lagrange Interpolation Formula

Although the ellipsoid found in Method B was far more reliable than Method A, I think this method is still not an accurate models of the jelly beans because there is some excess left over on the sides and coordinates plotted on the functions are still slightly off the boundary of the Jelly Bean because of its slightly irregular shape and deformities. Hence it still strays away from the 'perfect' model of a jelly bean. Therefore I decided to analytically find polynomials that will help model the top hemisphere of the jelly bean. These polynomial functions were found using the Lagrange Interpolation Formula. The Lagrange Interpolation formula helps derive a polynomial using arbitrary points. According to this formula, to model the 'best fit' for any unique polynomial of degree n, $n+1$ coordinate points are required. Let $(x_1, y_1) \dots (x_{n+1}, y_{n+1})$ be the $n+1$ points of different x-coordinates, polynomial $P(x)$ equals:

$$P(x) = \sum_{i=1}^{n+1}\left[y_i \prod_{J=1}^{n+1}\left(\frac{x - x_J}{x_i - x_J}\right)\right] = \sum_{i=1}^{n+1} \frac{(x - x_1) \dots (x - x_{i-1})(x - x_{i+1}) \dots (x - x_{n+1})}{(x_i - x_1) \dots (x_i - x_{i-1})(x_i - x_{i+1}) \dots (x_i - x_{n+1})} y_i \quad (Brilliant, 2020) \qquad (45)$$

where n is the degree of the polynomial, $P(x_i) = y_i$ and $i \in \{1, 2, \dots, n+1\}$ and $J \neq i$.

It is important to note that the greater the degree of the polynomial, the more accurate the final model will be. However, it is self-explanatory that using greater degrees (like 7^{th} and 8^{th} degrees), although more accurate, will be unnecessarily complicated and time-consuming to evaluate. Therefore to simplify the process I have decided to divide the jelly bean into three different sections, and will instead use 3 different quadratic equations ($Z(x)$ for each sub-section) to model the jelly bean.

Because the highest degree of a quadratic function is 2, according to the Lagrange Interpolation I will require 3 coordinates in each subsection to find the best fit in each part of the Jelly Bean. In order to apply the Lagrange Interpolation for a quadratic, $n = 2$ and $i = 3$ hence $Z(x)$ equals

$$Z(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} y_3 \text{ (“Lagrange Interpolating Polynomial.”)} \qquad (46)$$

Each sub-section of Jelly Bean is categorized into Right, Middle, Left for referencing purposes. I divided these subsections using two parallel lines (placed vertically on the graph see figure 15) where the jelly bean shows deformations (based on visual observations and previous models of functions). Mostly observed near $x = -6\ mm$ and $x = +6\ mm$.

Substituting the values of the coordinates into $Z(x)$ and evaluating this function will yield the final quadratic equation in its standard form $(ax^2 + bx + c)$ however this would be very time consuming. So instead I decided to expand $Z(x)$ and compare coefficients to find the value $a, b \text{ and } c$ in the standard quadratic equation where point (x_1, y_1); point (x_2, y_2); and point (x_3, y_3) represent the 3 coordinates in each subsection.

- $a = \frac{y_1}{(x_1-x_2)(x_1-x_3)} + \frac{y_2}{(x_2-x_1)(x_2-x_3)} + \frac{y_3}{(x_3-x_1)(x_3-x_2)}$ ("Lagrange Interpolating Polynomial.")
- $b = -\left[\frac{y_1x_2+y_1x_3}{(x_1-x_2)(x_1-x_3)} + \frac{y_2x_3+y_2x_1}{(x_2-x_1)(x_2-x_3)} + \frac{y_3x_1+y_3x_2}{(x_3-x_1)(x_3-x_2)}\right]$ ("Lagrange Interpolating Polynomial.")
- $c = x_1x_2x_3\left[\frac{1}{(x_1-x_2)(x_1-x_3)} + \frac{2}{(x_2-x_1)(x_2-x_3)} + \frac{1}{(x_3-x_1)(x_3-x_2)}\right]$ ("Lagrange Interpolating Polynomial.")

3 arbitrary points were identified on the Jelly Bean on each sub-section, see **figure 20** and table below and their coordinates were substituted into $Z(x)$ to find the quadratic function to model each subsection of the jelly bean using a quadratic to form a piece wise function for the jelly bean given by $L(x)$.

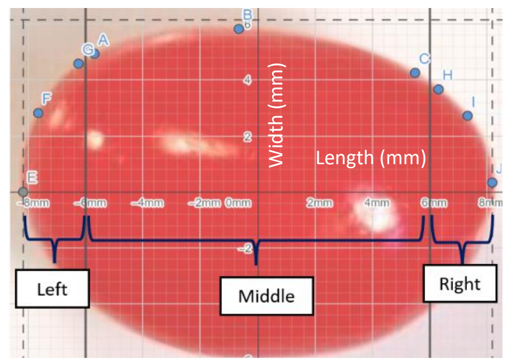

Figure 20 *Division of Jelly Bean into three sections: Left, Middle and Right (Candidate on GeoGebra, 2021)*

Table 6 Coordinates on Left Subsection of the Jelly Bean			**Table 7** Coordinates on Middle Subsection of the Jelly Bean			**Table 8** Coordinates on Right Subsection of the Jelly Bean		
i	x	y	i	x	y	i	x	y
1	-8.175	0	1	-5.691	4.944	1	6.302	3.669
2	-7.651	2.820	2	-0.659	5.826	2	7.315	2.722
3	-6.246	4.584	3	5.484	4.258	3	8.175	0.336

Table 9 Finding the Quadratic Functions using Lagrange Interpolation Formula

Subsection	Quadratic Function (after substituting the value of $a, b \text{ and } c$)	Domain (see graph)
Left	$-1.21419\,x^2 - 15.10470\,x - 42.33761$	$-8.175 < x < -6$
Middle	$-0.03853\,x^2 - 0.06935\,x + 5.79716$	$-6 < x < 6$
Right	$-0.98080\,x^2 + 12.41947\,x - 35.64544$	$6 < x < 8.175$

Using the quadratic equations in table 9, the jelly bean can be modelled by a piecewise function:

$$L(x) = \begin{cases} -1.21419\,x^2 - 15.10470\,x - 42.33761, & -8.175 < x < -6 \\ -0.03853\,x^2 - 0.06935\,x + 5.79716, & -6 < x < 6 \\ -0.98080x^2 + 12.41947\,x - 35.64544, & 6 < x < 8.175 \end{cases} \quad (48)$$

Using Lagrange Interpolation formula, I basically found the standard form of the quadratic equation for each of the three subsection to form a piece wise function that precisely modelled the jelly bean with a function which lied on the exact boundaries of the candy.

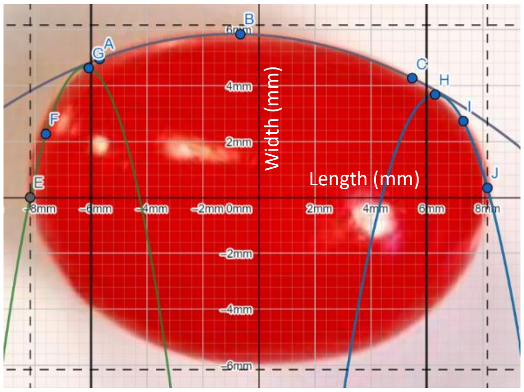

Figure 21 *Quadratic Functions for each Subsection using Lagrange Interpolation Formula on GeoGebra (Candidate, 2021)*

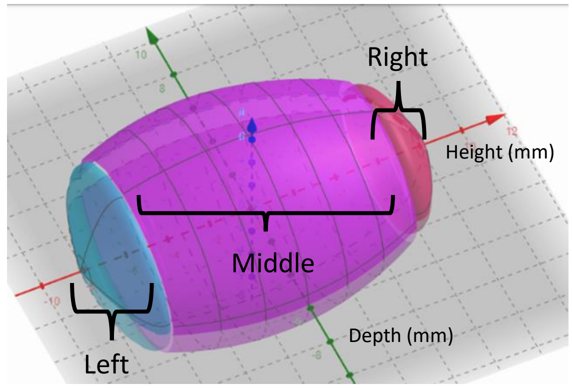

Figure 22 *3D Image of the 3 Quadratic Functions on GeoGebra (Candidate, 2021)*

Table 10 Finding the Volume of Revolution (for each subsection)

Subsection	Volume of Revolution Formula	Volume (mm^3)
Left	$\pi \int_{-8.175}^{-6} [-1.21419x^2 - 15.10470x - 42.33761]^2 \, dx$	85.213
Middle	$\pi \int_{-6}^{6} [-0.03853x^2 - 0.06935x + 5.79716]^2 \, dx$	1081.543
Right	$\pi \int_{6}^{8.175} [-0.98080x^2 + 12.41947x - 35.64544]^2 \, dx$	57.322
Total Volume		1224.078

Therefore the total no. of jelly beans predicted by this function is:

$$No. of\ Jelly\ Beans = \frac{V_O}{V_J} = \frac{1342084.746\ mm^3}{1224.058\ mm^3} = 1096.422 \approx 1096\ jelly\ beans \quad \mathbf{(49)}$$

Values for volume is quoted to 3 decimal places for consistency and accuracy. Final number of jelly beans is rounded to the nearest whole number because it is discrete.

VI. EVALUATION

After tediously counting the exact no. of jelly beans in the jar, there were 1068 jelly beans. To determine the accuracy of the models used aforementioned, the table below shows a comparison of the percentage error between each function used to model the shape of jelly beans:

$$Percentage\ error = \left| \frac{Literature\ Value - Obtained\ Value}{Literature\ Value} \right| \times 100 \quad \mathbf{(50)}$$

Table 11 Percentage Errors for all 3 Methods of Modelling

Method	No. of Jelly Beans	Percentage Error	Potential Cause of Error
Fourth Root Function	1142	8.55 %	• Excess Area uncovered by the modelling function • Sharp edges that accounted for extra area
Ellipsoid Function	981	6.74 %	• Excess area not enclosed by the function, hence a large deviation from the true value.
Lagrange Interpolation Formula	1096	4.18 %	• A polynomial of degree 2 is used, a higher degree or more division of subsections of the jelly bean would lead to a more accurate value. • Deformities may vary depending on the jelly bean

VII. CONCLUSION

Ultimately, the initial aim was successfully met. It can be concluded that this approach of modelling the jelly bean and using the volumes of revolution certainly does give a candidate a leg-up and more accurate results if modelled by an appropriate function whose coordinated lie exactly or as close as possible to the boundaries of the candy. This allows each participant to make an 'educated guess' to potentially win this game. It is also evident that improving the function used to model reduces the percentage error. Based on the results above, it is clear that using Lagrange transformation to model the shape of jelly beans proves to be the most efficacious method simply because it yields the smallest percentage error. Provided that candidates use an accurate function to model the shape of the candy, the candidate can maximize their estimate of the number is as close as 20% away from the true value of candies inside the jar.

To examine the success of using volumes of revolution as method to estimate the no. of jelly beans, I did another survey with a sample size (of 35 people randomly chosen): refer to **table 12** and taught them how to use volumes of revolution to predict the no of candies. I then asked them to make predictions based on their calculations and the table above shows their estimates.

Table 12 - Estimated No. of Candies						
923	942	821	874	924	792	834
730	786	849	876	945	923	835
744	825	711	954	984	883	864
920	900	825	742	895	736	693
756	873	986	844	974	942	913
MEAN ($\bar{x}$) of this Sample					857	
STANDARD DEVIATION (σ)					83	

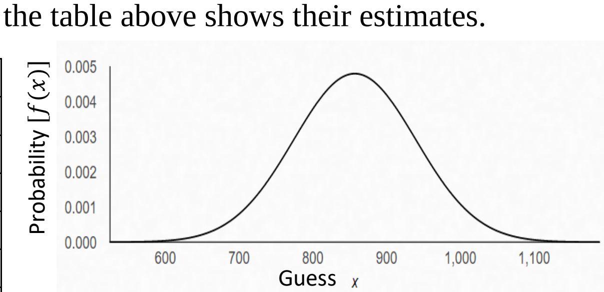

Figure 23 *Normal Distribution curve for estimates using Volumes of Revolution on GeoGebra (Candidate, 2021)*

Based on the current result of the volume of revolution, only one person was really close to the actual value who predicted that no. of candies inside that jar was 1015 (based on the survey conducted earlier). Although this person, like the 34 other people, took a wild guess based on his intuition, he would have likely won the competition. The standard of deviation is smaller the mean is more closer to the real value of the number of candies, hence volumes of revolution appears to be a more reliable method that taking a wild guess in candy guessing contests. My own estimate based on a random guess was far of the actual value. Hence, taking an ‘educated guess’ using this method is more logical and increases the competitiveness of the game.

VIII. REFLECTION

In general, I feel that my overall findings certainly provide a more reliable alternative to random guesses in candy games, and also makes the game more competitive however considering the intended target audience, and variety of candy guessing games I think the accuracy of this method of using volumes of revolution will be limited by the following extraneous factors:

No.	Limitations
1.	Although this method is more accurate and certainly provides you with a significant advantage over the other participants, with a really small margin of error of 20%, it is worth noting that this method is far complicated and difficult for the general audience, especially for younger years. Since candy-guessing games are the most popular among junior years, the efficiency of this method can be doubted for a young target audience.
2.	I also realized that the method of using modelling and volumes of revolution to predict the number of candies in jar only works for a uniform jar of candies (i.e. containing only a single type of candy). I have often hosted contests, with mixed jars of candies are used where there is mixture of smarties, skittles, Jelly beans, Sugar Coated candies and hard candies. In this case volume of revolution cannot be used because the percentage of each type of candy with in the jar is not known to the candidate.
3.	I also noticed that packing efficiency varies for different shapes and a small deviation in the Packing fraction has a large influence on the overall no. jelly beans estimated. But for certain types of jelly beans that do not resemble spheroids or are of a non-uniform shape it is simply impossible to determine the packing efficiency of the candy and hence this method will not work non-uniform candies.

However, despite the difficulties, assumptions and inaccuracies in modelling the shape, using volumes of revolution to estimate the no. of candies certainly makes your guess more close to the actual value and increases your chances of winning the game.

Plausible Extensions: Next time I can possibly investigate how optimize the packing efficiency of jelly beans and look at the possible arrangements inside a jar. In future I could also examine how much aluminium foil is wasted in wrapping chocolates and candies using surface area of revolution.

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Appendix A– Google Form Survey Screenshots

Survey Form on Google Forms (Monday 12th December 2021

Survey Mode: *Participant View*

Questions Responses Settings

Winter Souk Carnival Game: Candy Guessing

Hi Guys,

Welcome to the Candy Guessing Game at our annual Winter Souk. This year get a chance to win a AED 100.00 Noon Voucher at today's winter souk carnival game just by guessing the number of candies in the jar. We will be announcing 3 winners whom have the closest guess to the actual number in this jar. Please see the image above of jar filled with regular jelly beans. Can you guess the number of jelly beans in this 22 cm by 11 cm jar?

Also please answer the second question about your thought process honestly. Your results will be kept anonymous. No additional advantage or benefit will be given to any method you choose.

Candy Jar Dimension: 22 cm by 11 cm

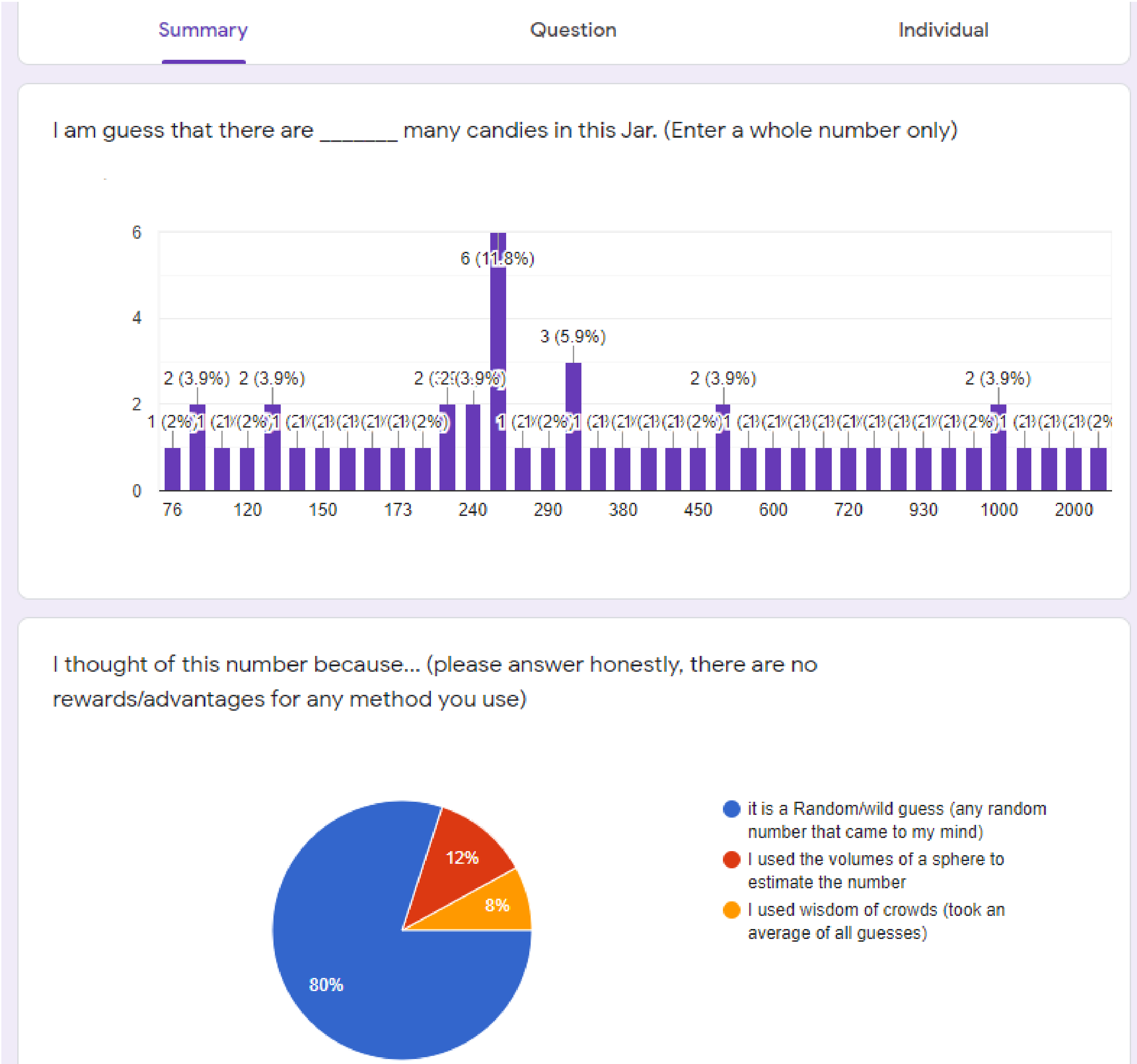
I am guess that there are ______ many candies in this Jar. (Enter a whole number only) *

Short answer text

I thought of this number because... (please answer honestly, there are no rewards/advantages for any method you use) *

- it is a Random/wild guess (any random number that came to my mind)
- I used the volumes of a sphere to estimate the number
- I used wisdom of crowds (took an average of all guesses)

Survey Mode: Owner View (Response Sheet)



Appendix B – Evidence of Personal Significance

Candidate organizing candy-guessing games using different brands and shapes of candies

(Taken: Tuesday 9th November 2018)

Candidate participating in Children's Carnival in 2020

(Taken: Tuesday 9th November 2020)